Multi-scaling Modelling in Financial Markets

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ABSTRACT

In the recent years, a new wave of interest spurred the involvement of complexity in finance which might provide a guideline to understand the mechanism of financial markets, and researchers with different backgrounds have made increasing contributions introducing new techniques and methodologies. In this paper, Markov-switching multifractal models (MSM) are briefly reviewed and the multi-scaling properties of different financial data are analyzed by computing the scaling exponents by means of the generalized Hurst exponent $H(q)$. In particular we have considered $H(q)$ for price data, absolute returns and squared returns of different empirical financial time series. We have computed $H(q)$ for the simulated data based on the MSM models with Binomial and Lognormal distributions of the volatility components. The results demonstrate the capacity of the multifractal (MF) models to capture the stylized facts in finance, and the ability of the generalized Hurst exponents approach to detect the scaling feature of financial time series.

1. INTRODUCTION

The concept of complex system has progressively become popular in economics, and recent developments suggest that non-traditional approaches, based on the tools of statistical and nonlinear physics, coupled with methods from computation intelligence, could provide novel ideas to direct studies in financial economics.[1] One of the most dynamic topics in this direction is the multi-scaling modelling in finance. As extended, Mandelbrot (1997)[2] translates the multi-scaling terminology from physics into finance by introducing the multifractal model of asset returns, and provides a fundamentally new class of stochastic processes in finance. The application of the scaling concept to financial markets has largely increased also in consequence of the abundance of available data.[3–5]

As one variant of multifractal (MF) processes, Markov-switching multifractal models (MSM) with simple specifications are revisited in this paper (Section 2); In order to see how well the estimated multifractal models capture the scaling property of financial time series, we estimate and compare the scaling exponents $H(q)$ by using the generalized Hurst exponent approach (reviewed in Section 3) for both empirical data and simulated data of the estimated MSM models with Binomial and Lognormal volatility components. Section 4 reports the empirical and simulation based results. A summary and concluding remarks are given in Section 5.

2. MARKOV-SWITCHING MULTIFRACTAL MODELS

Financial asset returns have traditionally been modelled based on the normal distribution. However, the empirical returns are characterized by stylized facts that imply non-normality. Multifractal processes in finance provide us with a new model with attractive stochastic properties, which takes into account the following stylized facts of financial markets: fat tails, volatility clustering, long-term dependence and multi-scaling.

However, the practical applicability of earlier versions of multifractal models suffers from the combinatorial nature and from its non-stationarity due to the restriction to a bounded interval. These limitations have been

Complex Systems II, edited by Derek Abbott, Tomaso Aste, Murray Batchelor, Robert Dewar,
Tiziana Di Matteo, Tony Guttmann, Proc. of SPIE Vol. 6802, 68021A, (2008)
0277-786X/08/$18 · doi: 10.1117/12.759585

overcome by introducing iterative versions of MF models, which preserve the multifractal and stochastic properties, making econometric analysis applicable. In the Markov-switching multifractal model, asset returns are modelled as:[6,7]

$$r_t = \sigma_t \cdot u_t \tag{1}$$

with innovations u_t drawn from the standard Normal distribution $N(0,1)$ and instantaneous volatility being determined by the product of k volatility components or multipliers $M_t^{(1)}$, $M_t^{(2)}$ ..., $M_t^{(k)}$ and a constant scale factor σ:

$$\sigma_t^2 = \sigma^2 \prod_{i=1}^{k} M_t^{(i)}, \tag{2}$$

Each volatility component is renewed at time t with probability γ_i depending on its rank within the hierarchy of multipliers and it remains unchanged with probability $1-\gamma_i$. The transition probabilities are specified by Calvet and Fisher[6] as:

$$\gamma_i = 1 - (1-\gamma_k)^{(b^{i-k})} \qquad\qquad i = 1, \dots k, \tag{3}$$

with parameters $\gamma_k \in [0,1]$ and $b \in (1,\infty)$. Different specifications of Eq. (3) can be arbitrarily imposed (cf.[7] and its earlier versions). By fixing $b = 2$ and $\gamma_k = 0.5$, we arrive to a relatively parsimonious specification:

$$\gamma_i = 1 - (0.5)^{(2^{i-k})} \qquad\qquad i = 1, \dots k. \tag{4}$$

For the choice of volatility components, a discrete version of MF process adopts the binomial distribution: $M_t^{(i)} \sim \{m_0,\ 2-m_0\}$ with $1 \le m_0 < 2$, and the binomial MF model is, therefore characterized by binomial random draws taking two discrete values with equal probability. Lux (2007) further introduces a continuous version of multi-fractal process by specifying the volatility components to be random draws from a Lognormal distribution (LN) with parameters λ and σ_m, i.e.

$$M_t^{(i)} \sim LN(-\lambda, \sigma_m^2). \tag{5}$$

In contrast to the combinatorial settings, (cf.[2,8]), a normalisation of the expectation value of $M_t^{(i)}$, that is, $E[M_t^{(i)}] = 1$ is imposed to prevent non-stationarity without assigning additional components, and it leads to

$$exp(-\lambda + 0.5\sigma_m^2) = 1 \qquad \Rightarrow \qquad \sigma_m = \sqrt{2\lambda}. \tag{6}$$

Note that the admissible parameter space for the location parameter λ is $\lambda \in [0,\infty)$ where in the borderline case $\lambda = 0$ the volatility process collapses to a constant (the same when $m_0 = 1$ in the Binomial case).

Using the iterative version of the multifractal model instead of its combinatorial predecessor and confining attention to unit time intervals, the resulting dynamics of Eq. (1) can also be seen as a particular version of a stochastic volatility model. With the rather parsimonious approach, this pertinent MF process preserves the hierarchical structure of MMAR while dispensing with its restriction to a bounded interval. In particular, the model captures some properties of financial time series, e.g. the power-law behaviour of the autocovariance function:

$$\langle (|r_t|^q - \langle |r_t|^q \rangle) \cdot (|r_{t+\tau}|^q - \langle |r_{t+\tau}|^q \rangle) \rangle \propto \tau^{2d(q)-1}, \tag{7}$$

for each qth moment and time lag τ, and $d(q)$ is a scaling function depending on q (for the detailed proof, cf.[6]). In contrast to other volatility models with long-term dependence,[9] MSM models allow for multi-scaling rather than uni-scaling with varying decay exponents for all powers of absolute values of returns.

3. GENERALIZED HURST EXPONENT

The scaling properties in time series have been studied in the literature by means of several techniques, such as the seminal work of Hurst (1951) on rescaled range[10] statistical analysis R/S and the modified R/S analysis of Lo (1991),[11] the multi-affine analysis (Peng et al., 1994),[12] the detrended fluctuation analysis,[13,14] etc. Scaling in data gives useful information on the underlying process, and the Hurst exponent analysis examines if some statistical properties of time series $X(t)$ (with $t = v, 2v, ..., T$) scale with the observation period (T) and the time resolution v. Such a scaling is characterized by an exponent H which is commonly associated with the long-term statistical dependence of the signal.

The generalized Hurst exponent (GHE) method aims to extend the traditional scaling exponent methodology, and this approach provides a natural, unbiased, statistically and computationally efficient estimator able to capture the scaling features of financial fluctuations.[15–17] It is essentially a tool to study directly the scaling properties of the data via the qth order moments of the distribution of the increments. The qth order moments appear to be less sensitive to the outliers than maxima/minima and different exponents q are associated with different characterizations of the multi-scaling behaviour of the signal $X(t)$. We consider the q-order moment of the distribution of the increments of a time series $X(t)$:

$$K_q(\tau) = \frac{\langle \mid X(t+\tau) - X(t) \mid^q \rangle}{\langle \mid X(t) \mid^q \rangle}, \tag{8}$$

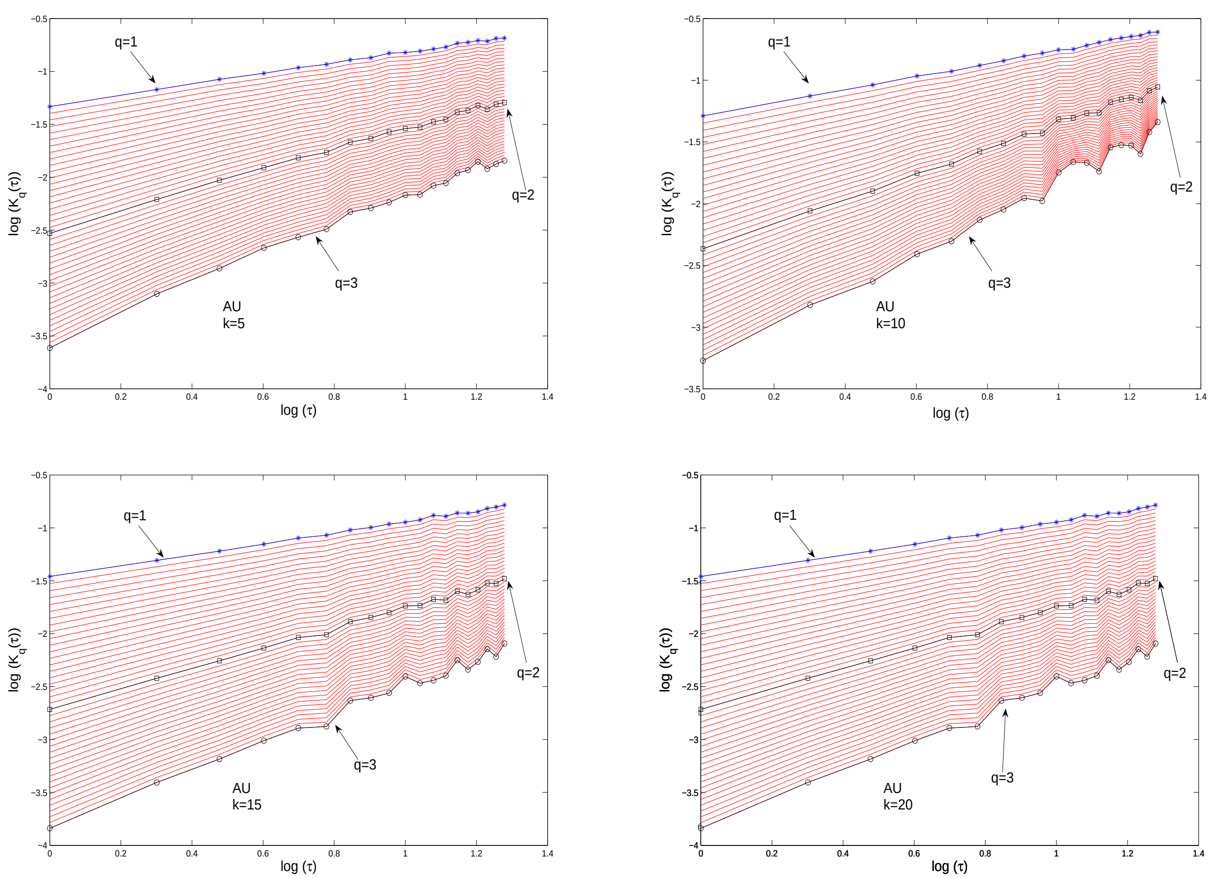

Figure 1. $K_q(\tau)$ as a function of τ on a loglog scale for AU with $X(t) = \log(p_t)$. Simulations are based on the Binomial model with $k = 5, 10, 15, 20$ (from up-left, up-right, low-left, low-right).

where the time interval τ varies between $v = 1$ day and $\tau_{\max}$ days. The generalized Hurst exponent $H(q)$ is then defined from the scaling behavior of $K_q(\tau)$, which can be assumed to follow the relation:

$$K_q(\tau) \sim \left(\frac{\tau}{v}\right)^{qH(q)}. \tag{9}$$

Within this framework, for $q = 1$, $H(1)$ describes the scaling behavior of the absolute values of the increments; for $q = 2$, $H(2)$ is associated with the scaling of the autocorrelation function.

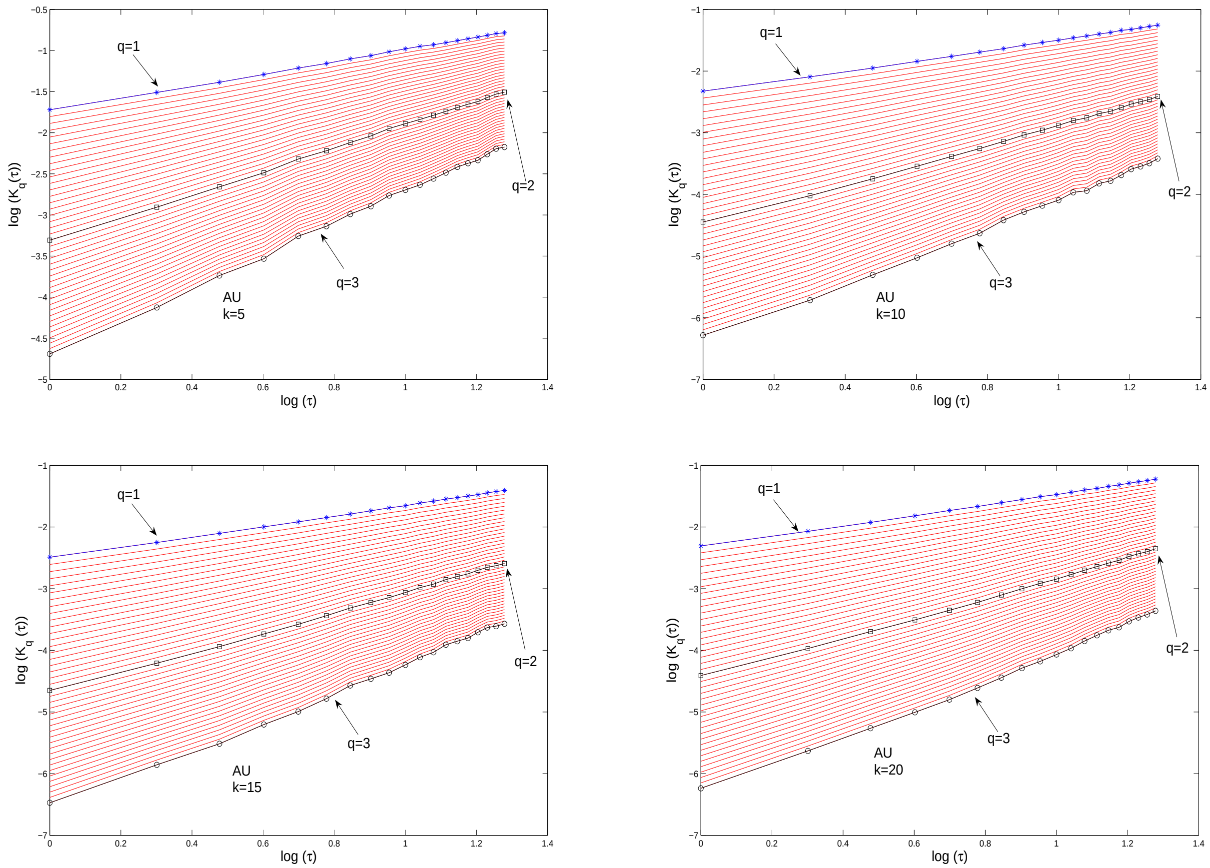

Figure 2. $K_q(\tau)$ as a function of τ on a loglog scale for AU with $X(t) = \sum_{t'=1}^{t} |r_{t'}|$. Simulations are based on the Binomial model with $k = 5, 10, 15, 20$ (from up-left, up-right, low-left, low-right).

4. RESULTS FROM EMPIRICAL AND SIMULATED STUDIES

In this paper, we consider daily data for foreign exchange rates of Australian Dollar to U.S. Dollar (AU) over the period from March 1973 to February 2004, and U.S. 1-year treasury constant maturity bond rates ($TB1$) in the period from June 1976 to October 2004. The daily prices are denoted as p_t, and returns are calculated as $r_t = \ln(p_t) - \ln(p_{t-1})$ for AU and as $r_t = p_t - p_{t-1}$ for $TB1$.

We have computed the q-order moments $K_q(\tau)$ (Eq. (8)) of $X(t) = \log(p_t)$ for AU and $X(t) = p_t$ for $TB1$ with τ in the range between $\tau = 1$ day and $\tau_{\max}$ days. We have verified its scaling behavior with τ, and the results for the empirical data and other type of data have been reported in.[18–20] In this paper, we present results for the simulated time series by means of Markov-switching multi-fractal (MSM) model[6,7] whose

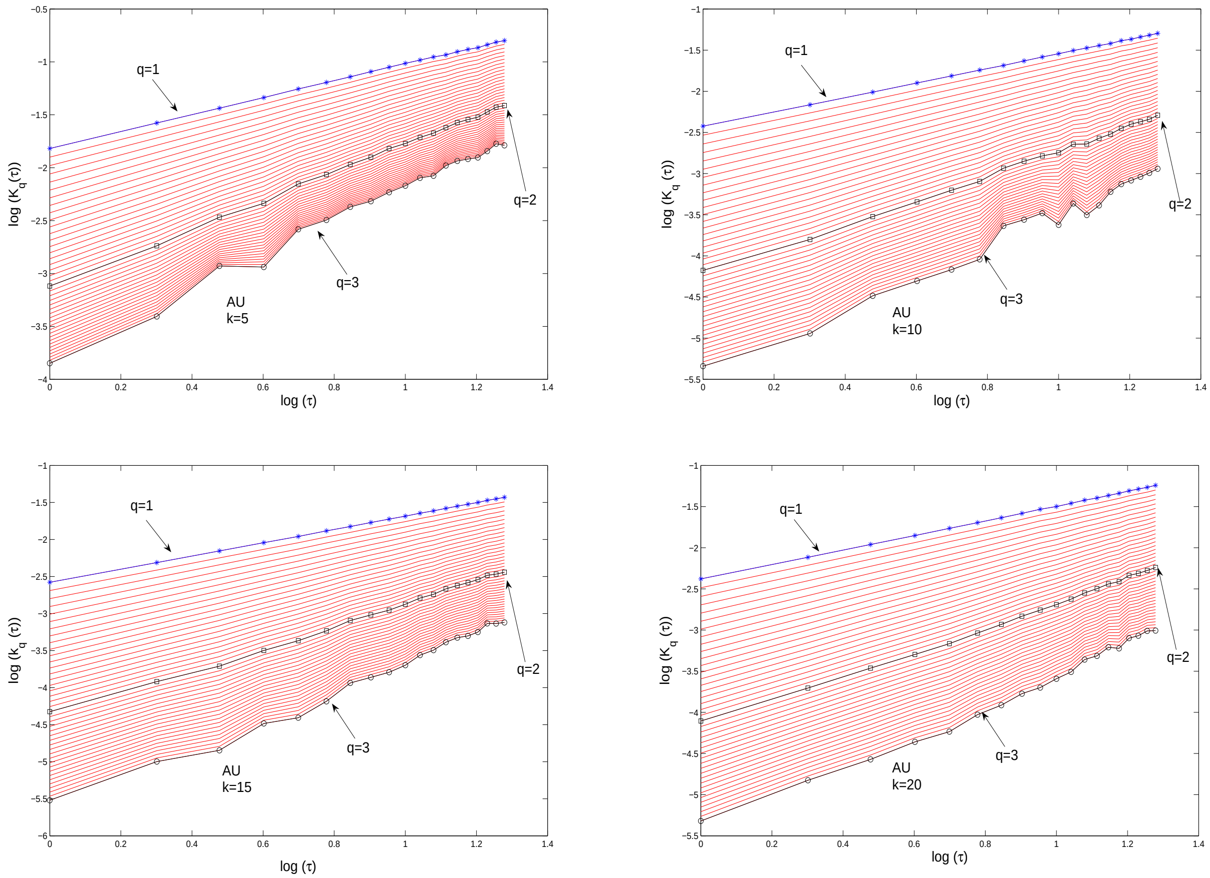

Figure 3. $K_q(\tau)$ as a function of τ on a loglog scale for AU with $X(t) = \sum_{t'=1}^{t} r_{t'}^2$. Simulations are based on the Binomial model with $k = 5, 10, 15, 20$ (from up-left, up-right, low-left, low-right).

parameters estimates have been done via Generalized Method of Moments (GMM) approach[7] for both discrete and continuous specifications of the volatility components: the Binomial and Lognormal distributions.[18–20]

In Figures 1 - 3, we report the $K_q(\tau)$ curves for the simulated time series of AU for $X(t) = \log(p_t)$, $X(t) = \sum_{t'=1}^{t} |r_{t'}|$ and $X(t) = \sum_{t'=1}^{t} r_{t'}^2$, respectively. Each graph within these figures corresponds to the Binomial model simulations with different values $k = 5, 10, 15, 20$. From these figures it emerges that Eq. (9) holds for all simulated time series with different $X(t)$. Similar scaling behaviors have been also found for $TB1$. Figure 4 presents the plots of simulated $K_q(\tau)$ curves based on the Lognormal model for AU and $TB1$, and we find similar behaviors with the ones based on the Binomial model. Binomial and Lognormal distributions for the volatility components produce similar results.

Figure 5 further reports the $\tau(q) = qH(q)$ as a function of q (Eq. (9)) for empirical time series and simulated data with $k = 5, 10, 15, 20$ based on the Binomial model; the left and right panels are corresponding to the case of $X(t) = \sum_{t'=1}^{t} |r_{t'}|$ for AU and $TB1$, respectively. We find that the curves are bending below their linear trend across each scenario, which verifies the multi-scaling property exhibited by the data. For small values of q the model gives a better agreement with the empirical curves. Moreover the case $k = 5$ seems to give good agreement in both cases. Similar studies with other financial time series have also been conducted and we skip them here for the brevity. We have also computed $H(1)$ and $H(2)$ from different empirical financial markets data for $X(t) = \log(p_t)$, as shown in Di Matteo et. al (2007).[20]

In order to test the robustness of our results, for each series we have analyzed the scaling properties varying τ_{max} between 5 and 19 days. We have computed the 99% confidence intervals of all the exponents using different

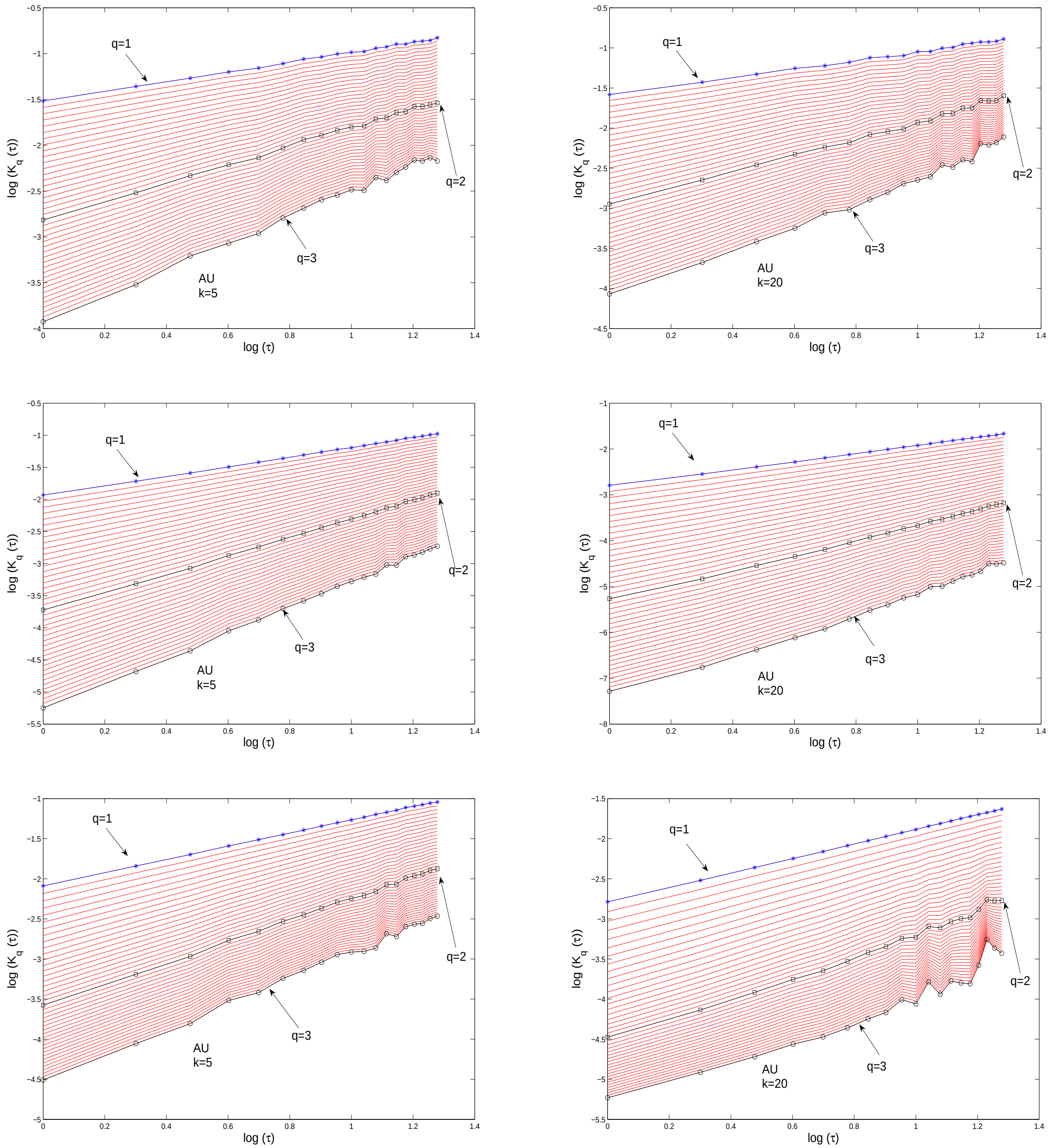

Figure 4. $K_q(\tau)$ as a function of τ on a loglog scale for AU. Simulations are based on the Lognormal model with $k = 5$ (left) and $k = 20$ (right) for the stochastic variable $X(t) = \log(p_t)$, $X(t) = \sum_{t'=1}^{t} |r_{t'}|$ and $X(t) = \sum_{t'=1}^{t} r_{t'}^2$ (from up to down).

τ_{max} values. The resulting exponents computed using different τ_{max} are stable in their values within a range of 10%. The empirical scaling exponents $H(1)$ and $H(2)$ are varying with different type of data, they are all different among each other and different from 0.5; The results are consistent with studies on high-frequency data.[4,16] $H(1)$ values based on simulated time series are different from 0.5, and they vary among different time series; $H(2)$ from simulated data are more homogenous and all close to 0.5 across different k. The fact that

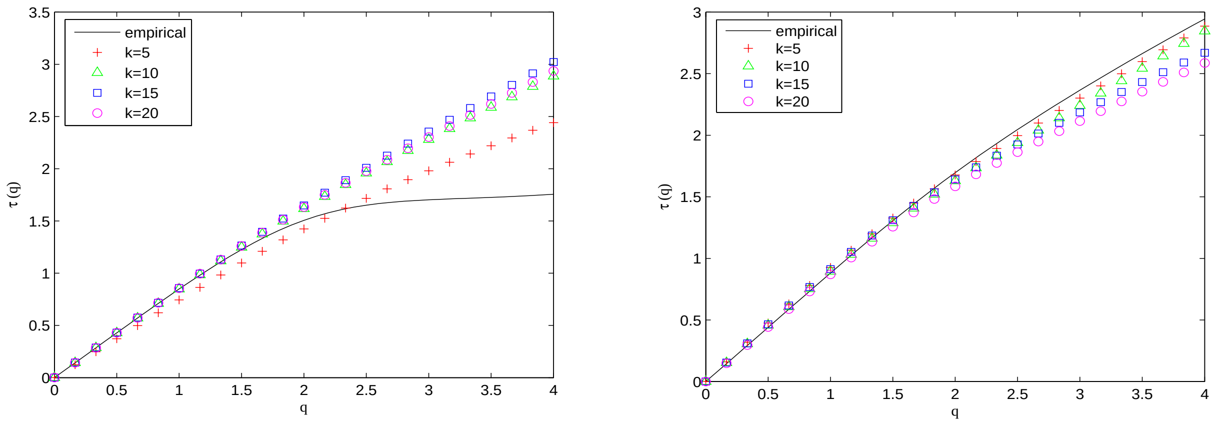

Figure 5. Plots of $\tau(q) = qH(q)$ as a function of q for empirical and simulated data with k = 5, 10, 15, 20. Simulations based on the Binomial model for AU (left) and $TB1$ (right) with $X(t) = \sum_{t'=1}^{t} |r_{t'}|$.

$H(2)$ from simulations for $X(t) = \log(p_t)$ does not change with k and is always equal to 0.5 might be explained by the fact that MSM model is reminiscent of the scaling property for the moments of absolute value of return (price changes) rather than the price p_t itself. Therefore, we have performed studies of GHE with the stochastic variable $X(t)$ in Eq. (8) to be $\sum_{t'=1}^{t} |r_{t'}|$ and we have found that the scaling exponent $H(2)$ in this case is varying across different financial assets and different k; both the empirical data and the simulated MSM models are characterized by estimates of $H(1)$ and $H(2)$ larger than 0.5; in particular there is a jump between the case $k = 5$ and others with larger k values for simulated time series. Finally we have also conducted comparisons by using MSM process with Lognormal specification, and there is not much difference observed for the results based on the Binomial and Lognormal models.

5. SUMMARIES AND CONCLUDING REMARKS

In this paper, we have reviewed the Markov-switching multifractal models (MSM) and the generalized Hurst exponent (GHE) approach. Based on the empirical estimates of MSM models with different financial markets data, we compared the scaling behavior of empirical and simulated data by using the generalized Hurst exponent approach. Our results show that MSM models are able to replicate the scaling property of absolute returns and squared returns, but not price data itself, and we have also found that there are very similar behaviors comparing the results from the models by using Binomial and Lognormal specifications of volatility components. By computing and comparing the scaling exponents for empirical and simulated data, the results show that the generalized Hurst exponent approach is a powerful tool to detect the scaling property of financial markets data for different stochastic variables, and it is a good tool to test the reliability of models such as MSM.

ACKNOWLEDGMENTS

T. Di Matteo acknowledges the partial support by ARC Discovery Projects: DP03440044 (2003) and DP0558183 (2005), COST P10 'Physics of Risk' project. The authors also would like to thank Prof. Thomas Lux for his help and many useful discussions.

REFERENCES

1. T. Lux, and M. Marchesi, "Scaling and criticality in a stochastic multi-agent model of a financial market," *Nature*, **397**, 498–500, 1999.
2. B. Mandelbrot, A. Fisher, and L. Calvet, "A Multifractal Model of Asset Returns," *Cowles Foundation for Research and Economics*, Manuscript, 1997.
3. U. A. Müller, M. M. Dacorogna, R. B. Olsen, O. V. Pictet, M. Schwarz, C. Morgenegg, "Statistical study of Foreign Exchange rates, empirical evidence of a price change scaling law, and intraday analysis," *Journal of Banking and Finance*, **14**, 1189-1208, 1990.
4. M. M. Dacorogna, R. Gençay, U. A. Müller, R. B. Olsen, O. V. Pictet, *An Introduction to High Frequency Finance*, **Academic Press**, San Diego, 2001.
5. T. Lux, "Detecting Multi-Fractal Properties in Asset Returns: An Assessment of the 'Scaling Estimator," *International Journal of Modern Physics*, **15**, 481–491, 2004.
6. L. Calvet, and A. Fisher, "How to forecast long-run volatility: regime switching and the estimation of multifractal processes," *Journal of Financial Econometrics*, **2**, 49–83, 2004.
7. T. Lux, "The Markow-switching multi-fractal model of asset returns: GMM estimation and linear forecasting of volatility," *Journal of Business and Economic Statistics*, **in Press**, 2007.
8. L. Calvet, and A. Fisher, "Multifractality in asset returns: theory and evidence," *Review of Economics and Statistics*, **84**, 381–406, 2002.
9. R. T. Baillie, T. Bollerslev, and H. Mikkelsen, "Fractionally Integrated Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, **74**, 3–30, 1996.
10. H. E. Hurst, "Long-term storage capacity of reservoirs," *Transaction of the American Society of Civil Engineers*, **116**, 770-808, 1951.
11. A. W. Lo, "Long-term memory in stock market prices," *Econometrica* **59**, 1279–1313, 1991.
12. C. K. Peng, S. V. Buldyrev, S. Havlin, M. Simons, H. E. Stanley, A. L. Goldberger, "Mosaic organization of DNA nucleodites," *Physical Review E*, **49**, 1685-1689, 1994.
13. M. Ausloos, "Statistical physics in foreign exchange currency and stock markets," *Physica A*, **285**, 48-65, 2000.
14. M. Bartolozzi, C. Mellen, T. Di Matteo, T. Aste, "Multi-scale correlations in different future markets," *European Physical Journal B*, **58**, 207-220, 2007.
15. T. Di Matteo, T. Aste, and M. Dacorogna, "Scaling behaviors in differently developed markets," *Physica A*, **324**, 183–188, 2003.
16. T. Di Matteo, T. Aste, and M. Dacorogna, "Long-term memories of developed and emerging markets: using the scaling analysis to characterize their stage of development," *Journal of Banking and Finance*, **29**, 827–851, 2005.
17. T. Di Matteo, "Multi-scaling in finance," *Quantitative Finance*, **7**, No. 1, 21–36, 2007.
18. R. Liu, T. Di Matteo, and T. Lux, "True and apparent scaling: The proximities of the markov-switching multifractal model to long-range dependence," *Physica A*, **383**, 35–42, 2007.
19. R. Liu, T. Di Matteo, and T. Lux, "Multifractality and long-range dependence of asset returns: The scaling behaviour of the markov-switching multifractal model with Lognormal volatility components," submitted for the proceeding of the Econophysics Colloquium 2007.
20. T. Di Matteo, R. Liu, and T. Lux, "A look at different scaling behaviors by means of the generalized Hurst exponent approach," in preparation, 2007.