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MULTIFRACTALITY AND LONG-RANGE DEPENDENCE OF ASSET RETURNS: THE SCALING BEHAVIOR OF THE MARKOV-SWITCHING MULTIFRACTAL MODEL WITH LOGNORMAL VOLATILITY COMPONENTS

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In this paper, we consider daily financial data from various sources (stock market indices, foreign exchange rates and bonds) and analyze their multiscaling properties by estimating the parameters of a Markov-switching multifractal (MSM) model with Lognormal volatility components. In order to see how well estimated models capture the temporal dependency of the empirical data, we estimate and compare (generalized) Hurst exponents for both empirical data and simulated MSM models. In general, the Lognormal MSM models generate "apparent" long memory in good agreement with empirical scaling provided that one uses sufficiently many volatility components. In comparison with a Binomial MSM specification [11], results are almost identical. This suggests that a parsimonious discrete specification is flexible enough and the gain from adopting the continuous Lognormal distribution is very limited.

Keywords: Markov-switching multifractal; scaling; Hurst exponent.

1. Introduction

The development of the multifractal approach goes back to Benoit Mandelbrot's work on turbulent flows [17]. Its adaptation for financial data resulted in the multifractal model of asset returns (MMAR) [18], which provides a new time series model with attractive stochastic properties accounting for the stylized facts of financial returns. However, the practical applicability of the MMAR suffers from its combinatorial nature and from its nonstationarity due to the restriction to a bounded interval. In addition, it suffers from a lack of applicable statistical methods [14,18]. These limitations have been overcome by the introduction of iterative versions of multifractal processes [4,15] which preserve the multifractal and stochastic properties of the earlier combinatorial models to a large extent but have more convenient asymptotical properties.

In this paper, we expand on our previous work [11] and compare the temporal scaling properties of the empirical data to those of estimated Markov-switching multifractal (MSM) models. Based on the empirical estimates obtained via the generalized method of moments (GMM), we conduct simulations of multifractal models and compare the empirical data and simulated ones in terms of their autocorrelation functions (ACFs). In addition, we compute Hurst exponents and perform explicit tests for long memory (temporal scaling), using two refinements of the traditional Hurst approach. One of these is the so-called generalized Hurst exponent, $H(q)$, for the qth order moment proposed in Refs. 7–9. The second is Lo's modified R/S approach [12], which allows one to adjust the rescaled range for possible short-memory effects. We proceed by comparing the scaling exponents for empirical data and simulated time series based on our estimated MSM models. The structure of the paper is as follows. In Sec. 2 we introduce the multifractal models. Section 3 reports the empirical and simulation-based results. A summary and concluding remarks are given in Sec. 4.

2. Markov-Switching Multifractal Models

In the MSM model, financial asset returns are modeled as

$$r_t = \sigma_t \cdot u_t \,, \tag{1}$$

with innovations u_t drawn from the standard normal distribution $N(0,1)$ and instantaneous volatility being determined by the product of k volatility components or multipliers $M_t^{(1)}$, $M_t^{(2)}, \ldots, M_t^{(k)}$ and a constant scale factor σ:

$$\sigma_t^2 = \sigma^2 \prod_{i=1}^{k} M_t^{(i)} . \tag{2}$$

Each volatility component is renewed at time t with probability γ_i, depending on its rank within the hierarchy of multipliers, and it remains unchanged with probability $1 - \gamma_i$. The transition probabilities are specified by Calvet and Fisher [4] as

$$\gamma_i = 1 - (1 - \gamma_k)^{(b^{i-k})}, \quad i = 1, \ldots, k \,, \tag{3}$$

with parameters $\gamma_k \in [0,1]$ and $b \in (1,\infty)$. Different specifications of Eq. (3) have been imposed (see Ref. 15 and its earlier versions). By fixing $b = 2$ and $\gamma_k = 0.5$, we arrive at a relatively parsimonious specification:

$$\gamma_i = 1 - \left(\frac{1}{2}\right)^{(2^{i-k})}, \quad i = 1, \ldots, k\,. \tag{4}$$

For the choice of volatility components, a popular version of the multifractal (henceforth MF) process adopts the Binomial distribution: $M_t^{(i)} \sim \{m_0, 2 - m_0\}$, with $1 \leq m_0 < 2$. Another prominent variant [15, 18] is the continuous version of the multifractal process that assumes the volatility components to be random draws from a Lognormal distribution[a] (LN) with parameters λ and σ_m, i.e.

$$M_t^{(i)} \sim \mathrm{LN}(-\lambda, \sigma_m^2)\,. \tag{5}$$

In line with the combinatorial settings [3,18], a normalization of the expectation of $M_t^{(i)}$, i.e. $E[M_t^{(i)}] = 1$, is imposed which leads to a restriction on the parameters of the Lognormal distribution:

$$\exp(-\lambda + 0.5\sigma_m^2) = 1 \Rightarrow \sigma_m = \sqrt{2\lambda}\,. \tag{6}$$

Note that the admissible parameter space for the location parameter λ is $[0,\infty)$, where in the borderline case $\lambda = 0$ the volatility process collapses to a constant (the same when $m_0 = 1$ in the Binomial case).

The above MF processes can be viewed as a special case of a Markov-switching process which makes maximum likelihood (ML) estimation feasible if the distribution of volatility components is discrete. In the Binomial case, state spaces are finite, so that maximum likelihood estimation is possible [4]. However, the applicability of ML encounters an upper bound for the number of cascade levels (about $k \leq 10$) because of the necessity to evaluate the $2^k \times 2^k$ transition matrix for every realization. The limits of current computational capability are reached with about 10 cascade levels. A more fundamental limitation is the restriction to cases that have discrete distributions of volatility components. Since MF processes with continuous distributions (such as the Lognormal distribution) of the volatility components imply an infinite number of states, ML is not applicable to them. Lux [15] proposed the generalized method of moments (GMM) approach as an alternative, which relaxes these computational restrictions and can be used in the case of the Binomial MF model for larger numbers of cascade levels ($k > 10$), and the Lognormal MF process. The exact algorithm and the analytical moment conditions for implementing the GMM (both the Binomial and the continuous Lognormal model) can be found in Ref. 15.

Using the iterative version of the MF model instead of its combinatorial predecessor and confining attention to unit time intervals, the resulting dynamics of

[a] The Lognormal distribution has the probability density function $f(x,\lambda,\sigma_m) = \frac{e^{-(\ln(x)-\lambda)^2/(2\sigma_m^2)}}{x\cdot\sigma_m\sqrt{2\pi}}$.

Eq. (1) can also be viewed as a particular version of a stochastic volatility model. The model also captures some of the most ubiquitous properties of financial time series, namely outliers (extreme realizations), volatility clustering and the power law behavior of the autocovariance function (see Refs. 6 and 16 for an overview of the stylized facts and scaling laws of financial data):

$$\langle(|r_t|^q - \langle|r_t|^q\rangle) \cdot (|r_{t+\tau}|^q - \langle|r_{t+\tau}|^q\rangle)\rangle \propto \tau^{2d(q)-1}, \tag{7}$$

where, for each qth moment and time lag τ, $d(q)$ is the pertinent scaling function depending on q. Although models of this class are partially motivated by empirical findings on long-term dependence of volatility, they do not obey the traditional definition of long memory, i.e. asymptotic power law behavior of autocovariance functions in the limit $t \to \infty$ or divergence of the spectral density at zero [1]. The iterative MF model is rather characterized by only "apparent" long memory with an asymptotic hyperbolic decline of the autocorrelation of absolute powers over a finite horizon, and an exponential decline thereafter. In the case of the MSM process, the approximately hyperbolic decline, therefore, holds only over the interval $1 \ll \tau \ll 2^k$ (for the detailed proof, see Ref. 4). In view of this preasymptotic temporal scaling, it seems interesting to explore how far the MF model is capable of reproducing the empirical results on the scaling laws of returns and their higher moments. This is the question we pursue in the present paper and its companion [11]. As it turns out, taking empirical Hurst exponents as "stylized facts," both the estimated Binomial and Lognormal MF models are capable of reproducing empirical statistics if enough volatility components are allowed for (i.e. for high k).

3. Comparison of Empirical and Simulated Series

In this paper, we consider daily data for a selection of stock exchange indices — the Dow Jones Composite 65 Average Index (*Dow*) and the NIKKEI 225 Average Index (*Nik*) over the period from January 1969 to October 2004; foreign exchange rates — British pound to US dollar (*UK*), and Australian dollar to US dollar (*AU*), over the period from March 1973 to February 2004; and US one-year and two-year treasury bond rates with constant maturity ($TB1$ and $TB2$, respectively) over the period from June 1976 to October 2004. The daily prices are denoted as p_t, and returns are calculated as $r_t = \ln(p_t) - \ln(p_{t-1})$.[b]

We estimate the Lognormal model parameters via the GMM. Table 1 presents the empirical estimates of the Lognormal model for various hypothetical numbers of cascade levels ($k = 5, 10, 15, 20$) using the same analytical moments as in Ref. 15 (the numbers within the parentheses are the standard errors). The pertinent estimates for the Binomial case have been reported in Ref. 11. For each time series, we find that the estimates for $k \geq 10$ are almost identical. In fact, analytical moment

[b]The US one- and two-year treasury constant maturity rates have been converted to equivalent bond prices before calculating returns.

Table 1. GMM estimates of the MSM model for different values of k.

		$k = 5$		$k = 10$		$k = 15$		$k = 20$	
	T	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$
Dow	9372	0.148	0.983	0.139	0.983	0.139	0.983	0.139	0.983
		(0.018)	(0.052)	(0.018)	(0.052)	(0.018)	(0.053)	(0.018)	(0.052)
Nik	9372	0.289	0.991	0.279	0.990	0.280	0.991	0.280	0.990
		(0.022)	(0.036)	(0.021)	(0.036)	(0.022)	(0.036)	(0.022)	(0.036)
UK	8493	0.096	1.053	0.079	1.058	0.078	1.058	0.078	1.058
		(0.018)	(0.027)	(0.017)	(0.026)	(0.017)	(0.027)	(0.017)	(0.027)
AU	8493	0.140	1.012	0.121	1.014	0.120	1.015	0.120	1.014
		(0.024)	(0.065)	(0.023)	(0.065)	(0.023)	(0.066)	(0.023)	(0.065)
*TB*1	7110	0.314	1.022	0.262	1.055	0.260	1.056	0.260	1.056
		(0.019)	(0.061)	(0.016)	(0.060)	(0.016)	(0.060)	(0.016)	(0.059)
*TB*2	7110	0.446	0.999	0.367	1.036	0.364	1.037	0.364	1.037
		(0.019)	(0.051)	(0.016)	(0.049)	(0.016)	(0.049)	(0.016)	(0.049)

Note: Estimation is based on the Lognormal model. T is the number of observations. All data had been standardized before estimation.

conditions in Ref. 15 show that higher cascade levels make a smaller and smaller contribution to the moments so that their numerical values would stay almost constant. If one monitors the development of estimated parameters with increasing k, one finds strong variations initially with a pronounced decrease of the estimates, which becomes slower and slower until eventually a constant value is reached somewhere around $k = 10$ for each time series.

As a prelude to our Monte Carlo study comparing the empirical and simulated scaling exponents, we plot the autocorrelation functions (ACFs) of absolute returns for empirical and simulated time series with different numbers of multipliers k (Fig. 1). We find that the simulated time series with $k = 5$ exhibits much faster decay than the empirical data. In contrast, the ones with larger values of k show the ability of the MSM model to replicate the apparently hyperbolic decay of the empirical ACF, namely the hyperbolic decay of the ACF. Recalling Eq. (7), we recognize that the approximately hyperbolic decline holds in the interval $1 \ll \tau \ll 2^k$, and therefore a MF process with a higher number of cascade levels implies a longer power law range of the autocorrelations, which means a larger region of apparent long-term dependence. We have also studied the ACFs based on the Binomial model, and they show pretty similar patterns [11]. Studies of other time series have also been pursued. We omit them here, as they are qualitatively similar to that of the Dow Jones index.

Since findings on temporal scaling have sometimes been disputed in the literature and various sources for apparent, spurious scaling have been identified (see Ref. 16 for an overview), it might be a worthwhile exercise to assess empirical scaling on the base of different algorithms. It also is the best practice in statistics and econometrics to use a battery of tests rather than relying on only one particular algorithm. We, therefore, compute $H(q)$ by the generalized Hurst exponent (GHE)

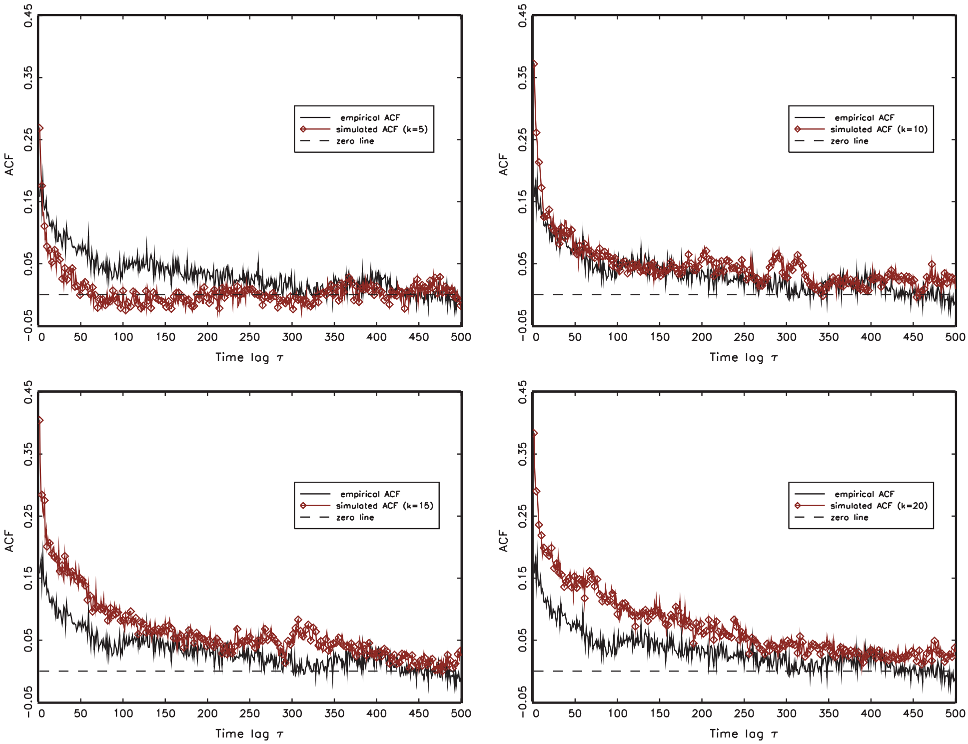

Fig. 1. Autocorrelation function (ACF) for the Dow Jones index and simulated time series (absolute returns). All simulations are based on the Lognormal model with different k values.

approach [7–9] and by the modified R/S method [12] for the same data sets, and we proceed by comparing the scaling exponents obtained for empirical data and simulated time series based on the estimated Lognormal MSM models.

We start with the GHE, which extends the traditional scaling exponent methodology. According to previous research, this approach provides a natural, unbiased, statistically and computationally efficient estimator able to capture very well the scaling features of financial fluctuations [7, 8]. It is essentially a tool for studying directly the scaling properties of the data via the qth order moments of the distribution of the increments. The qth order moments appear to be less sensitive to the outliers than maxima/minima, and different exponents q are associated with different characterizations of the multiscaling behavior of the signal $X(t)$. We consider the q-order moment of the distribution of the increments (with $t = v, 2v, \ldots, T$) of a time series $X(t)$: $K_q(\tau) = \frac{\langle |X(t+\tau) - X(t)|^q \rangle}{\langle |X(t)|^q \rangle}$, where the time interval τ varies between $v = 1$ day and $\tau_{\max}$ days. The GHE $H(q)$ is then defined from the scaling behavior of $K_q(\tau)$, which can be assumed to follow the relation $K_q(\tau) \sim (\frac{\tau}{v})^{qH(q)}$. Within this framework, for $q = 1$, $H(1)$ describes the scaling behavior of the absolute values of the increments. When $q = 2$, $H(2)$ is associated with the scaling of the ACF.

Table 2. $H(1)$ and $H(2)$ for the empirical and simulated data.

	$H(1)$					$H(2)$				
	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Dow	0.684	0.748	0.847	0.867	0.866	0.709	**0.710**	0.797	0.812	0.810
	(0.034)	(0.010)	(0.017)	(0.023)	(0.025)	(0.027)	(0.011)	(0.017)	(0.020)	(0.021)
NIK	0.788	**0.804**	0.895	0.909	0.901	0.753	**0.739**	0.811	0.818	0.813
	(0.023)	(0.010)	(0.015)	(0.019)	(0.030)	(0.021)	(0.014)	(0.019)	(0.021)	(0.025)
UK	0.749	0.707	0.797	0.820	0.821	0.735	0.681	**0.765**	0.783	0.784
	(0.023)	(0.010)	(0.019)	(0.027)	(0.027)	(0.026)	(0.011)	(0.019)	(0.024)	(0.024)
AU	0.827	0.742	**0.836**	**0.857**	**0.857**	0.722	**0.706**	0.790	0.806	0.807
	(0.017)	(0.010)	(0.019)	(0.024)	(0.025)	(0.024)	(0.012)	(0.019)	(0.021)	(0.022)
*TB*1	0.842	0.773	**0.876**	0.896	0.898	0.807	0.696	**0.785**	**0.799**	**0.800**
	(0.023)	(0.011)	(0.017)	(0.022)	(0.022)	(0.027)	(0.016)	(0.022)	(0.024)	(0.023)
*TB*2	0.771	**0.800**	0.895	0.913	0.913	0.761	0.697	**0.778**	**0.788**	**0.791**
	(0.025)	(0.011)	(0.016)	(0.020)	(0.020)	(0.029)	(0.018)	(0.026)	(0.027)	(0.027)

Note: Emp refers to the empirical estimates of $H(1)$ and $H(2)$. $k = 5$, $k = 10$, $k = 15$ and $k = 20$ refer to the mean and standard deviation of the exponent values based on 1000 simulated time series with pertinent k (Lognormal model). Bold numbers show those cases for which we cannot reject the identity of the Hurst coefficients obtained for empirical and simulated data, i.e. the empirical exponents fall into the range between the 2.5 and 97.5 percent quantile of the simulated data.

For the results reported in Table 2, we focus on $H(q)$ for $q = 1$ and $q = 2$. We present the pertinent estimates for both the empirical time series and the mean and standard deviation of 1000 simulated time series for each set of estimated parameters. The values for $H(1)$ and $H(2)$ are averages computed from a set of log regressions of $K_q(\tau)$ against τ for different $\tau_{\max}$ (between 5 and 19 days). The underlying stochastic variable $X(t)$ in Refs. 7–9 is defined as the sum of absolute values of returns, $X(t) = \sum_{t'=1}^{t} |r_{t'}|$. The second and seventh columns of Table 2 report the empirical $H(1)$ and $H(2)$, and values in the other columns are the mean values over the corresponding 1000 simulations for different k values: $5, 10, 15, 20$, with errors given by their standard deviations. Boldface numbers show those cases which fail to reject the null hypothesis that the mean of the GHEs from the simulations equals the empirical GHE at the 5% level based on the distribution of our 1000 Monte Carlo samples. We find that the exponents from the simulated time series vary across different cascade levels k. In particular, for the stock market indices, we find coincidence between the empirical series and simulation results for the scaling exponents $H(1)$ and $H(2)$ for the Nikkei index, and $H(2)$ for the Dow Jones index only for $k = 5$. For the exchange rate data, only $H(2)$ for *UK* fails to reject the null hypothesis when $k = 10$, whereas the estimated coefficients $H(1)$ and $H(2)$ are significantly different from the simulated ones in all other cases. We also observe that the simulations successfully replicate the empirical measurements of *AU* for $H(1)$ when $k = 10, 15, 20$. In the case of the US bond rates *TB*1 and *TB*2, we find good agreement for $H(2)$ when $k = 10, 15, 20$. While the empirical numbers are in nice agreement with previous results in Refs. 7–9, it is interesting to note that simulated data with $k \geq 10$ have a tendency towards even higher estimated Hurst coefficients than found in the pertinent empirical records. Similar studies for the Binomial model show almost identical numerical results for simulated data for all time series and at all k. This underscores the impression from previous studies [11, 15] that the empirical performance of both types of MF models is virtually identical.

An alternative approach to estimation of H is the well-known rescaled range algorithm, dating back to Hurst (1951). It uses the range of a time series x_t:

$$R_T = \max_{1 \leq t \leq T} \sum_{t=1}^{T} (x_t - \bar{x}) - \min_{1 \leq t \leq T} \sum_{t=1}^{T} (x_t - \bar{x}) \,, \tag{8}$$

with $\bar{x}$ the estimate of the mean. If R is rescaled by the sample standard derivation, it obeys temporal scaling with the same exponent H as the original time series. In the traditional R/S algorithm, H is also extracted via a log regression of R/S versus t according to the asymptotic scaling relationship $(R/S)_t \sim \alpha t^H$. Note that H provides an alternative estimator to the above GHE at $q = 1$ for the scaling of the range and related quantities. The spectrum of results obtained for H in Table 5 shows how much this estimate can be affected by different levels of hypothesized short-term dependence. Note also that $H(1) = H(2) = H$ would hold for unifractal time series, but only $H(1) = H$ and $H(1) \neq H(2)$ would apply for multifractal processes. Unfortunately, the original R/S statistic is biased in the presence of short-run autocorrelation. Lo developed a modified R/S statistic that is more immune to the presence of short-run dependence and allows an explicit test of the hypothesis of long-run dependence ($H \neq 0.5$) conditional on the maximum extent of short-run autocorrelation allowed for [12]. Lo's adjustment for short-run autocorrelations uses the Newey–West heteroskedasticity and autocorrelation consistent estimates instead of the sample standard derivation. Since the statistic $Q_l = T^{-0.5} \cdot R_T/S_l$ (S_l is the Newey–West estimator with l lags considered to cover short-run dependence) converges to the range of a Brownian bridge, the null hypothesis of absence of long memory can be tested explicitly. The original estimator of Hurst is obtained by setting $l = 0$.

To shed further light on the ability of the MF model to replicate empirical scaling behavior, we also performed calculations using this modified rescaled range analysis, whose results are reported in Tables 3–5. Table 3 presents Lo's test statistics for both empirical data and 1000 simulated time series (absolute returns) based on the Lognormal model with different values of k and for different truncation lags, $l = 0, 5, 10, 25, 50, 100$. We find that the values are varying with different truncation lags and, more specifically, that they are monotonically decreasing for both the empirical and simulation-based statistics. Table 4 reports the number of rejections of the null hypothesis of short-range dependence based on 95% and 99% confidence levels. The rejection numbers for each single k are decreasing as the truncation lag l increases, but the proportion of rejections remains relatively high for higher cascade levels, $k = 10, 15, 20$. The modified R/S approach would quite reliably

Table 3. Lo's R/S statistics for the empirical and simulated data.

	$l = 0$					$l = 5$					$l = 10$				
	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Dow	3.005	1.710	4.959	6.511	6.515	2.661	1.475	3.956	5.025	5.116	2.427	1.369	3.484	4.380	4.459
		(0.380)	(1.316)	(1.773)	(1.830)		(0.327)	(1.021)	(1.338)	(1.375)		(0.302)	(0.884)	(1.140)	(1.167)
Nik	7.698	1.837	4.633	5.809	5.802	6.509	1.530	3.616	4.495	4.487	5.836	1.405	3.175	3.925	3.917
		(0.428)	(1.207)	(1.543)	(1.528)		(0.352)	(0.926)	(1.168)	(1.157)		(0.321)	(0.802)	(1.004)	(0.996)
UK	6.821	1.570	4.480	5.995	6.035	5.912	1.390	3.712	4.874	4.900	5.333	1.304	3.313	4.298	4.319
		(0.377)	(1.237)	(1.837)	(1.842)		(0.332)	(0.988)	(1.410)	(1.417)		(0.309)	(0.863)	(1.202)	(1.211)
AU	7.698	1.670	4.832	6.247	6.294	6.731	1.443	3.891	4.941	4.975	6.103	1.341	3.436	4.316	4.344
		(0.380)	(1.300)	(1.852)	(1.818)		(0.325)	(1.017)	(1.401)	(1.366)		(0.300)	(0.882)	(1.192)	(1.159)
*TB*1	8.611	1.499	3.954	5.130	5.156	6.975	1.303	3.188	4.077	4.095	6.018	1.220	2.829	3.587	3.599
		(0.353)	(1.019)	(1.355)	(1.386)		(0.302)	(0.798)	(1.043)	(1.064)		(0.281)	(0.696)	(0.901)	(0.917)
*TB*2	6.616	1.474	3.717	4.548	4.654	5.607	1.277	3.009	3.647	3.724	4.958	1.196	2.684	3.233	3.295
		(0.341)	(0.978)	(1.213)	(1.243)		(0.292)	(0.774)	(0.942)	(0.969)		(0.271)	(0.678)	(0.818)	(0.841)

Table 3. (*Continued*)

	$l = 25$					$l = 50$					$l = 100$				
	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Dow	2.042	1.228	2.809	3.467	3.528	1.736	1.145	2.334	2.829	2.875	1.464	1.092	1.930	2.287	2.320
		(0.269)	(0.690)	(0.865)	(0.880)		(0.246)	(0.555)	(0.676)	(0.683)		(0.229)	(0.439)	(0.517)	(0.516)
Nik	4.760	1.250	2.569	3.142	3.134	3.941	1.161	2.153	2.600	2.592	3.220	1.106	1.802	2.137	2.130
		(0.280)	(0.631)	(0.778)	(0.775)		(0.255)	(0.513)	(0.619)	(0.619)		(0.236)	(0.412)	(0.482)	(0.481)
UK	4.348	1.185	2.710	3.439	3.454	3.575	1.112	2.266	2.816	2.825	2.871	1.065	1.880	2.274	2.280
		(0.279)	(0.678)	(0.906)	(0.914)		(0.258)	(0.544)	(0.701)	(0.708)		(0.241)	(0.427)	(0.527)	(0.533)
AU	5.035	1.205	2.779	3.421	3.445	4.130	1.123	2.311	2.790	2.811	3.281	1.071	1.911	2.251	2.269
		(0.265)	(0.689)	(0.900)	(0.870)		(0.242)	(0.553)	(0.697)	(0.673)		(0.224)	(0.437)	(0.525)	(0.508)
*TB*1	4.536	1.116	2.318	2.889	2.896	3.492	1.060	1.958	2.397	2.400	2.640	1.029	1.652	1.978	1.976
		(0.253)	(0.549)	(0.700)	(0.707)		(0.235)	(0.446)	(0.556)	(0.559)		(0.221)	(0.355)	(0.430)	(0.429)
*TB*2	3.896	1.097	2.223	2.645	2.687	3.100	1.044	1.897	2.227	2.257	2.408	1.016	1.619	1.864	1.886
		(0.244)	(0.540)	(0.641)	(0.660)		(0.228)	(0.440)	(0.516)	(0.531)		(0.216)	(0.352)	(0.406)	(0.415)

Note: Emp stands for the empirical Lo's statistics, and $k = 5$, $k = 10$, $k = 15$ and $k = 20$ refer to the mean and standard deviation of Lo's statistics based on the corresponding 1000 simulated time series with pertinent k (Lognormal model).

Table 4. Number of rejections for Lo's R/S statistics.

	$l = 0$								$l = 5$								$l = 10$							
	$k = 5$		$k = 10$		$k = 15$		$k = 20$		$k = 5$		$k = 10$		$k = 15$		$k = 20$		$k = 5$		$k = 10$		$k = 15$		$k = 20$	
	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡
Dow	324	154	999	998	1000	1000	1000	1000	121	47	996	986	1000	999	1000	1000	66	16	986	955	999	991	998	992
Nik	444	250	999	996	1000	998	1000	1000	172	67	990	971	996	991	998	995	86	30	970	926	989	978	995	979
UK	191	108	999	994	999	998	1000	1000	106	37	985	971	997	992	1000	997	60	15	973	943	992	981	997	988
AU	276	137	1000	999	1000	1000	1000	1000	114	36	998	984	998	993	1000	995	61	14	984	952	993	985	996	993
*TB*1	153	69	997	985	998	993	1000	998	50	13	970	929	991	975	996	986	27	5	929	849	974	959	986	971
*TB*2	124	57	995	972	997	994	999	992	38	9	943	872	985	961	984	963	18	3	878	796	965	921	964	919
	$l = 25$								$l = 50$								$l = 100$							
	$k = 5$		$k = 10$		$k = 15$		$k = 20$		$k = 5$		$k = 10$		$k = 15$		$k = 20$		$k = 5$		$k = 10$		$k = 15$		$k = 20$	
	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡
Dow	18	2	923	852	983	943	981	960	7	1	795	640	919	845	934	870	1	1	542	341	784	628	793	653
Nik	27	4	864	754	961	915	958	917	8	1	688	506	892	766	880	755	1	1	415	250	689	518	681	512
UK	16	4	907	802	971	937	977	936	8	1	752	586	905	834	915	840	4	1	496	300	760	618	764	602
AU	15	3	918	835	967	933	984	959	5	1	771	616	902	823	928	833	1	1	517	335	754	606	762	612
*TB*1	6	1	779	623	920	862	941	863	2	1	559	366	811	686	817	676	1	1	266	113	604	404	592	379
*TB*2	4	1	729	567	877	776	878	788	1	1	508	314	739	590	762	617	1	1	242	97	512	300	528	316

Note: $k = 5$, $k = 10$, $k = 15$ and $k = 20$ refer to the number of rejections at 95% (†) and 99% (‡) confidence levels (these intervals are given by [0.809, 1.862] and [0.721, 2.098], respectively) for the 1000 simulated time series.

Table 5. Lo's modified R/S Hurst exponent H values for the empirical and simulated data.

	$l = 0$					$l = 5$					$l = 10$				
	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Dow	0.620	0.556	**0.671**	**0.699**	**0.702**	0.607	0.540	**0.647**	**0.672**	**0.674**	0.597	0.532	**0.633**	**0.658**	**0.660**
		(0.024)	(0.030)	(0.032)	(0.032)		(0.024)	(0.029)	(0.031)	(0.030)		(0.024)	(0.028)	(0.030)	(0.030)
Nik	0.723	0.564	0.664	**0.688**	**0.688**	0.705	0.544	0.637	**0.660**	**0.660**	0.693	0.534	0.623	**0.646**	**0.646**
		(0.025)	(0.029)	(0.031)	(0.030)		(0.025)	(0.029)	(0.030)	(0.029)		(0.025)	(0.028)	(0.029)	(0.029)
UK	0.712	0.547	**0.662**	**0.693**	**0.693**	0.696	0.533	0.641	**0.670**	**0.671**	0.685	0.526	0.629	**0.657**	**0.657**
		(0.026)	(0.031)	(0.035)	(0.035)		(0.026)	(0.030)	(0.033)	(0.033)		(0.026)	(0.029)	(0.032)	(0.032)
AU	0.726	0.554	0.670	**0.697**	**0.699**	0.711	0.538	0.646	**0.672**	**0.673**	0.700	0.530	0.633	**0.657**	**0.658**
		(0.025)	(0.030)	(0.034)	(0.033)		(0.025)	(0.029)	(0.033)	(0.031)		(0.024)	(0.029)	(0.032)	(0.031)
*TB*1	0.743	0.543	0.651	**0.680**	**0.681**	0.719	0.527	0.627	**0.654**	**0.655**	0.702	0.520	0.614	**0.640**	**0.641**
		(0.026)	(0.030)	(0.032)	(0.031)		(0.026)	(0.029)	(0.031)	(0.030)		(0.025)	(0.028)	(0.030)	(0.030)
*TB*2	0.713	0.541	0.644	**0.667**	**0.669**	0.694	0.525	0.620	**0.642**	**0.644**	0.681	0.517	0.608	**0.629**	**0.631**
		(0.026)	(0.030)	(0.031)	(0.032)		(0.025)	(0.030)	(0.030)	(0.031)		(0.025)	(0.029)	(0.030)	(0.030)

Table 5. (*Continued*)

	$l = 25$					$l = 50$					$l = 100$				
	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Dow	0.578	0.520	**0.610**	**0.632**	**0.634**	0.560	0.512	**0.590**	**0.610**	**0.612**	0.542	0.507	**0.569**	**0.587**	**0.589**
		(0.024)	(0.028)	(0.029)	(0.028)		(0.023)	(0.027)	(0.027)	(0.027)		(0.023)	(0.026)	(0.026)	(0.025)
Nik	0.671	0.522	0.600	**0.622**	**0.621**	0.650	0.514	0.581	**0.601**	**0.602**	0.628	0.509	0.561	**0.580**	**0.580**
		(0.025)	(0.028)	(0.028)	(0.028)		(0.024)	(0.027)	(0.027)	(0.027)		(0.023)	(0.026)	(0.026)	(0.026)
UK	0.662	0.516	0.607	**0.632**	**0.633**	0.641	0.509	0.587	**0.611**	**0.611**	0.617	0.504	0.567	**0.588**	**0.588**
		(0.025)	(0.028)	(0.031)	(0.030)		(0.025)	(0.027)	(0.029)	(0.029)		(0.024)	(0.026)	(0.027)	(0.027)
AU	0.679	0.518	0.610	**0.632**	**0.633**	0.657	0.510	0.589	**0.610**	**0.611**	0.631	0.505	0.569	**0.586**	**0.588**
		(0.024)	(0.028)	(0.031)	(0.029)		(0.024)	(0.027)	(0.029)	(0.028)		(0.023)	(0.026)	(0.027)	(0.026)
*TB*1	0.670	0.510	0.592	**0.616**	**0.616**	0.641	0.504	0.573	**0.595**	**0.596**	0.609	0.501	0.554	**0.574**	**0.574**
		(0.025)	(0.027)	(0.029)	(0.028)		(0.025)	(0.026)	(0.028)	(0.027)		(0.024)	(0.025)	(0.026)	(0.025)
*TB*2	0.653	0.508	0.587	**0.606**	**0.608**	0.628	0.502	0.569	**0.587**	**0.588**	0.597	0.499	0.552	**0.567**	**0.569**
		(0.025)	(0.028)	(0.029)	(0.029)		(0.024)	(0.027)	(0.027)	(0.028)		(0.024)	(0.025)	(0.026)	(0.026)

Note: Emp stands for the empirical value of Lo's Hurst exponent, and $k = 5$, $k = 10$, $k = 15$ and $k = 20$ refer to the mean and standard deviation of Lo's Hurst exponent based on the corresponding 1000 simulated time series with different k. Boldface numbers show those cases in which empirical Hs fall into the corresponding 2.5–97.5 percent quantiles of the 1000 simulation-based values of H.

reject the null hypothesis of long memory for $k = 5$, but in most cases it would be unable to do so for higher numbers of volatility components, even if we allow for large truncation lags up to $l = 100$. This appears to be in harmony with the impression conveyed by Fig. 1. The corresponding Hurst exponents are given in Table 5. The empirical values of H are decreasing when l increases, and a similar behavior is observed for the simulation-based H for given values of k. The decrease of H with higher l is well known from other studies [2, 5, 13]. It is explained by removal of short-run dependence with higher lag lengths l that would otherwise be attributed to the long-memory estimate H. In the limit with $l \to \infty$, H should converge to 0.5. Of course, l might be misspecified and, if chosen too high, might bias the estimate of H downward. In empirical research, various automatic lag selection schemes can be used for optimal data-driven selection of l. Here we are interested in the whole pattern of results and the comparison of empirical and simulated data from this perspective. We also find that the Hurst exponent values are increasing with increasing cascade level k for given l. Boldface numbers show those cases which fail to reject the null hypothesis that the mean of the simulation-based Hurst exponent equals the empirical Hurst exponent at the 5% level, and we observe similar scenarios for the pertinent results based on the Binomial model reported in Ref. 11. There are significant jumps between the values of $k = 5$ and $k = 10$ as in previous tables, and we observe good overall agreement between the empirical and simulated data for practically all series for $k \geq 10$, but not so for the MSM models with a smaller number of volatility components, e.g. $k = 5$.

It is also interesting to note that even for the original Hurst estimate obtained in the absence of short-run effects ($l = 0$), the numbers differ quite strongly between the two methods we have used to assess temporal scaling. Also in terms of replicability of empirical results, we find quite pronounced differences. In particular, we are never able to reject the identity of the empirical and simulated Hurst exponents under Lo's modified R/S approach if we allow for a sufficiently large number of MF components. In contrast, results for the generalized $H(q)$ algorithm are much more mixed. However, noting the monotonic increase of simulated Hurst coefficients in Table 2 with increasing k, we might actually be able to find fitting intermediate numbers of multipliers for which coincidence of empirical and simulated scaling could be obtained. This might allow us to be somewhat more sanguine about these results. We, therefore, feel that we can safely conclude that the MF model is able to replicate empirical measurements of temporal scaling (be it true or apparent) relatively closely. Whether the different behavior of both algorithms arises from the lack of the $H(q)$ method to account for short-term dependence, or from its higher sensitivity to only apparent scaling, would be an interesting topic to pursue in future research.

4. Summary and Concluding Remarks

In this paper, we have investigated the scaling behavior of estimated Markov-switching multifractal models with Lognormal volatility components. Based on the empirical estimates via the GMM, we have studied simulated time series and compared their autocorrelation functions with the ones from empirical data. In addition to these qualitative comparisons, we have calculated the empirical and simulated scaling exponents by using the generalized Hurst exponent and the modified R/S approaches. Comparing the results from the Lognormal model to our previous study on the Binomial model [11], we observe that there is not much difference between these discrete and continuous versions of multifractal processes. This finding is also in line with the very similar goodness-of-fit and forecasting performance of MSM models reported in Ref. 15. Our results also demonstrate that typically MSM models with a relatively large number of volatility components ($k \geq 10$) are required to capture the long-term dependence of absolute values of returns. Since we know that the iterative Markov-switching MF models have only preasymptotic (i.e. apparent or spurious, strictly speaking) scaling, these results also show that we can replicate the stylized facts without "true" asymptotic scaling. Since the preasymptotic region can be arbitrarily large, this difference between true and apparent asymptotics may be of little practical concern.

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References

[1] Beran, J., *Statistics for Long-Memory Processes* (Chapman & Hall, New York, 1994).
[2] Brooks, C., A measure of persistence in daily pound exchange rates, *Appl. Econ. Lett.* **2** (1995) 428–431.
[3] Calvet, L. and Fisher, A., Multi-fractality in asset returns: Theory and evidence, *Rev. Econ. Stat.* **84** (2002) 381–406.
[4] Calvet, L. and Fisher, A., How to forecast long-run volatility: Regime switching and the estimation of multifractal processes, *J. Fin. Econometrics* **2** (2004) 49–83.
[5] Crato, N. and De Lima, P. J. F., Long-range dependence in the conditional variance of stocks returns, *Econ. Lett.* **45** (1994) 281–285.
[6] Dacorogna, M. M., Gençay, R., Müller, U. A., Olsen, R. B. and Pictet, O. V., *An Introduction to High Frequency Finance* (Academic Press, San Diego, 2001).
[7] Di Matteo, T., Aste, T. and Dacorogna, M., Scaling behaviors in differently developed markets, *Physica A* **324** (2003) 183–188.
[8] Di Matteo, T., Aste, T. and Dacorogna, M., Long term memories of developed and emerging markets: Using the scaling analysis to characterize their stage of development, *J. Bank. Fin.* **29** (2005) 827–851.
[9] Di Matteo, T., Multi-scaling in finance, *Quant. Fin.* **7**(1) (2007) 21–36.

[10] Hurst, H., Long term storage capacity of reservoirs, *Trans. Am. Soc. Civil Eng.* **116** (1951) 770–808.
[11] Liu, R., Di Matteo, T. and Lux, T., True and apparent scaling: The proximities of the Markov-switching multifractal model to long-range dependence, *Physica A* **383** (2007) 35–42.
[12] Lo, A. W., Long-term memory in stock market prices, *Econometrica* **59** (1991) 1279–1313.
[13] Lux, T., Long-term stochastic dependence in financial prices: Evidence from the German stock market, *Appl. Econ. Lett.* **3** (1996) 701–706.
[14] Lux, T., Detecting multi-fractal properties in asset returns: The failure of the "scaling estimator," *Int. J. Mod. Phys.* **15** (2004) 481–491.
[15] Lux, T., The Markov-switching multi-fractal model of asset returns: GMM estimation and linear forecasting of volatility, *J. Bus. Econ. Stat.* **26** (2008) 194–210.
[16] Lux, T. and Ausloos, M., Market fluctuations I: Scaling, multi-scaling and their possible origins, in: Bunde, A., Kropp, J. and Schellnhuber, H.-J. (eds.), *The Science of Disasters: Climate Disruptions, Heart Attacks and Market Crashes* (Springer, Berlin, 2002).
[17] Mandelbrot, B., Intermittent turbulence in self similar cascades: Divergence of high moments and dimension of carrier, *J. Fluid Mech.* **62** (1974) 331–358.
[18] Mandelbrot, B., Fisher, A. and Calvet, L., *Cowles Foundation Discussion Papers 1164–1166*, available at http://www.ssrn.com (Yale University, 1997).